Solution-Mid-Exam 2011-2012

(1). (i) If X is the union of path-connected open sets U_α each containing the basepoint $x_0 \in X$ and if each intersection $U_\alpha \cap U_\beta$ is path-connected, then the homomorphism

$$\Phi : *_\alpha \pi_1(U_\alpha) \to \pi_1(X)$$

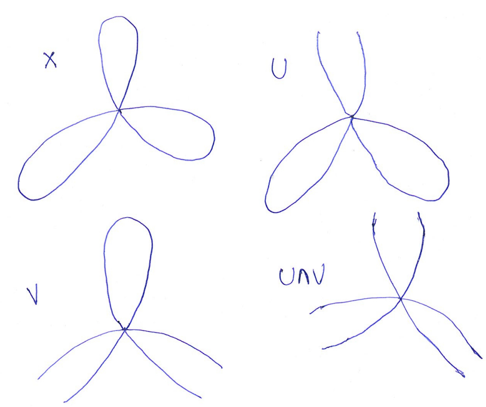
is surjective. If in addition each intersection $U_\alpha \cap U_\beta \cap U_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form

$$i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$$

for $\omega \in \pi_1(U_\alpha \cap U_\beta)$, where $i_{ab} : \pi_1(U_a \cap U_b) \to \pi_1(U_a)$ is the homomorphism induced by the inclusion $i : U_a \cap U_b \hookrightarrow U_a$, and hence Φ induces an isomorphism

$$\pi_1(X) \cong *_\alpha \pi_1(U_\alpha)/N.$$

(ii) We will use the Seifert-van Kampen Theorem to calculate the fundamental group. Let U, $V \subseteq X$ be as pictured (with the end points being open).



Since $U \cap V$ is contractible, then its fundamental group is trivial. this makes our calculation easier since we get that our normal subgroup N from the theorem is also trivial. U is homotopy equivalent to the figure eight, so

$$\pi_1(U, x_0) = \mathbb{Z} * \mathbb{Z}.$$

Also, V is homotopy equivalent to $\mathbb{S}^1$, so $\pi_1(V, x_0) = \mathbb{Z}$. This tells us that the pushout of U and V is $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ (since $*$ is associative). Thus π_1 of the 3-bouquet of circles is $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

To generalize we simply set U and V to be the as above, where V is still homotopy equivalent to $\mathbb{S}^1$, but U is homotopy equivalent to the $(n-1)$-bouquet of circles. This still gives us that $U \cap V$ is contractible, so by induction the fundamental group of the n-bouquet is $\mathbb{Z} * ... * \mathbb{Z}$ (n times).

We can also show that $\mathbb{Z} * ... * \mathbb{Z}$ (n times) is the free group on n generators, Denote F_n. We first use the universal property of a free group. Let Y be the set $\{x_1, x_2, ..., x_n\}$ and define $f : X \to \mathbb{Z} * ... * \mathbb{Z}$ (n times) as $f(x_i) = z_i$ where z_i is the generator from the ith copy of $\mathbb{Z}$. Since F_n is free we have that there exist a homomorphism $g : F_n \to \mathbb{Z} * ... * \mathbb{Z}$ (n times) such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & F_n \arrow[d, "g"] \\ & \mathbb{Z} * ... * \mathbb{Z} \end{tikzcd}$$

or $f = g \circ i$. Also g is bijective on the generators of each group, so it is bijective on the entire sets. Now define

$$f_1 : \pi_1(U, x_0) = \mathbb{Z} * ... * \mathbb{Z} \ (n-1 \ times) \to F_n$$

as $f_1(z_i) = x_i$ for $i = 1$,..., $n-1$ where the x_is are generators of F_n. also define

$$f_2 : \pi_1(V, x_0) = \mathbb{Z} \to F_n$$

with $f_2(1) = x_n$. Since $\mathbb{Z} * ... * \mathbb{Z}$ (n *times*) is a pushout then there exist a homomorphism

$$h : \mathbb{Z} * ... * \mathbb{Z} \ (n \ times) \to F_n$$

such that $h(\pi_1(j_1)) = f_1$ and $h(\pi_1(j_2)) = f_2$, where $j_1 : U \to X$ and $j_2 : V \to X$ are inclusions map. On the generators,

$$g \circ h(z_i) = g(x_i) = z_i$$

and

$$h \circ g(x_i) = h(z_i) = x_i,$$

So g and h are inverses. Therefore

$$\mathbb{Z} * \dots * \mathbb{Z} \ (n \ times) \cong F_n.$$

(iii) Let $X \subseteq \mathbb{R}^3$ be the union of m lines through the origin. Consider the homotopy:

$$f_t(x) = (1-t)x + t\frac{x}{|x|}.$$

Then $f_t : \mathbb{R}^3 - X \to \mathbb{R}^3 - X$ defines $\mathbb{S}^2 - \{x_1, x_2, ..., x_{2m}\}$ as a deformation retract of $\mathbb{R}^3 - X$, where $x_1, \ . \ . \ .x_{2m}$ are the $2m$ points in the intersection $X \cap \mathbb{S}^2$. Therefore

$$\pi_1(\mathbb{R}^3 - X) \cong \pi_1(\mathbb{S}^2 - \{x_1, x_2, ..., x_{2m}\}).$$

$\mathbb{S}^2$ without k points is homeomorphic to $\mathbb{R}^2$ without $k-1$ points, and this space has the homotopy type of the wedge sum of $k-1$ copies of $\mathbb{S}^1$. Hence: $\pi_1(\mathbb{R}^3 - X) \cong \pi_1(\mathbb{S}^2 - \{x_1, x_2, ..., x_{2m}\}) \cong \pi_1(\mathbb{R}^2 - \{y_1, y_2, ..., y_{2m-1}\}) \cong \pi_1(\mathbb{S}^1 \vee ... \vee \mathbb{S}^1) \cong \mathbb{Z} * ... * \mathbb{Z} \ (2n-1 \ times)$.

(2). (i) Two covering projections, $p_i : \overline{X}_i \to X$, $i = 1, 2$, are said to be equivalent if there is a homeomorphism $f : \overline{X}_1 \to \overline{X}_2$ such that $p_2 \circ f = p_1$.

(ii) Given two covering projections $p : E \to B$ and $p : E' \to B'$ with $p(e_0) = p'(e'_0) = b_0$, suppose $f : E \to E'$ is an equivalence such that $f(e_0) = e'_0$. Then, since f is a homeomorphism, we have

$$f_*(\pi_1(E, e_0)) = \pi_1(E'.e'_0).$$

Therefore,

$$p_*(\pi_1(E, e_0)) = p'_*(\pi_1(E', e'_0)).$$

Conversely, suppose

$$p_*(\pi_1(E, e_0)) = \eta^{-1} p'_*(\pi_1(E', e'_0))\eta,$$

for some element $\eta \in \pi_1(B, b_0)$. Let λ be a path in E' such that $\lambda(0) = e'_0$, $[p' \circ \lambda] = \eta$ and $\widetilde{e'_0} = \lambda(1)$.Then we have

$$[\lambda]^{-1}\pi_1(E', e'_0)[\lambda] = \pi_1(E', \widetilde{e'_0}).$$

Therefore, it follows that

$$p'_*(\pi_1(E', \widetilde{e'_0})) = \eta^{-1} p_*(\pi_1(E', e'_0))\eta = p_*(\pi_1(E, e_0)).$$

By applying the lifting criterion, either way, we get maps $f : E \to E'$ and $g : E' \to E$ such that $p' \circ f = p$ and $p \circ g = p'$ and $f(e_0) = \widetilde{e'_0}$, $g(\widetilde{e'_0}) = e_0$. Now $p' \circ f \circ g = p$ and $f \circ g(\widetilde{e'_0}) = \widetilde{e'_0}$. Therefore, by Unique Lifting Property, we have $f \circ g = id_{E'}$. Likewise, we see $g \circ f = id_E$. Therefore f (and g) defines an equivalence of p and p'.

(iii) The subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by two elements $(m, 0)$ and $(0, n)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, so we might guess that the covering space is $T = \mathbb{S}^1 \times \mathbb{S}^1$ itself. By taking the covering map

$$z_1 \times z_2 \to z_1^m \times z_2^n,$$

we see that the generators corresponding to $(1, 0)$ and $(0, 1)$ in $\pi_1(T)$ map to $m \times 0$ and $0 \times n$, respectively. Thus this is the correct covering space.

(3). (i) Let $p : \overline{X} \to X$ be a covering map. We call $T : \overline{X} \to \overline{X}$ a covering transformation if (i) T is a homeomorphism and (ii) $p = p \circ T$. The set of all covering transformations forms a group under function composition, which is called the automorphism (Deck transformation) group $Cov(\overline{X}/X)$.

(ii) For $[\alpha] \in N(H)$ and $\bar{y} \in E$ we define $\phi([\alpha])(\bar{y})$ as follows: choose any continuous path $f : [0, 1] \to E$ with $f(0) = e_0$ and $f(1) = \bar{y}$. Let $\widetilde{\alpha} : [0, 1] \to E$ be the lift of $\alpha : [0, 1] \to X$ with $\widetilde{\alpha}(0) = e_0$ and let $f' : [0, 1] \to E$ be the lift of $p \circ f : [0, 1] \to B$ with $f'(0) = \widetilde{\alpha}(1)$. We define

$$\phi([\alpha])(\bar{y}) = f'(1).$$

We have to show that

(a) ϕ is well-defined;

(b) $\phi([\alpha]) \in Cov(E/B)$;

(c) ϕ is a homomorphism;

(d) ϕ is onto;

(e) $\ker\phi = H$.

(a): We wish to show that the definition of ϕ is independent of the choice of f. To this end, let $g : [0,1] \to E$ be another continuous path with $g(0) = e_0$ and $g(1) = \bar{y}$. It is our goal to show that $f'(1) = g'(1)$. Put $\bar{z} = \widetilde{\alpha}(1)$, then

$$[\widetilde{\alpha}]^{-1}\pi_1(E, e_0)[\widetilde{\alpha}] = \pi_1(E, \bar{z}).$$

Applying p_* to this equation, we get

$$[\alpha]^{-1}p_*(\pi_1(E, e_0))[\alpha] = p_*(\pi_1(E, \bar{z})).$$

However, by assumption,

$$[\alpha]^{-1}p_*(\pi_1(E, e_0))[\alpha] = p_*(\pi_1(E, e_0)).$$

so that

$$p_*(\pi_1(E, \bar{z})) = p_*(\pi_1(E, e_0)).$$

This means that the elements of $\pi_1(B, b_0)$ which lift to loops at e_0 are the same as those which lift to loops at $\bar{z}$. Consequently,

$$[p \circ (f.g^{-1})] \in p_*(\pi_1(E, e_0)) = p_*(\pi_1(E, \bar{z})).$$

In other words,

$$[p \circ (f.g^{-1})] = [p \circ \gamma]$$

for some $\gamma : [0,1] \to E$ with $\gamma(0) = \gamma(1) = \bar{z}$. From Unique Lifting Property, we have that $(p \circ f)(p \circ g^{-1})$ lifts to some loop at $\bar{z}$. Since f' and g' are the unique lifts of $p \circ f$ and $p \circ g$ at $\bar{z}$, respectively, we must have $f'(1) = g'(1)$.

(b): First of all note that, by definition, we have

$$p \circ \phi([\alpha])(\bar{y}) = p(f'(1)) = p(f(1)) = p(\bar{y}).$$

If N is an elementary neighborhood of $y = p(\bar{y})$ and U_1 and U_2 the path components of $p^{-1}(N)$ containing $\bar{y}$ and $\phi([\alpha])(\bar{y})$, respectively, we get $\phi([\alpha])(U_1) = U_2$. To see this, all we have to do is choose the path f in the definition of $\phi([\alpha])(\bar{w})$ for $\bar{w} \in U_1$ to always begin with the same path f running from $f(0) = e_0$ to $f(1) = \bar{y}$ and concatenate it with a path h running from $h(0) = \bar{y}$ to $h(1) = \bar{w}$ which stays in U_1. This observation yields continuity of $\phi([\alpha])$. Also, it is clear that $\phi([\alpha]^{-1})$ is the inverse of $\phi([\alpha])$. In summary,

$\phi([\alpha]) \in Cov(E/B)$.

(c): Let $[\alpha]$, $[\beta] \in N(H)$ and $\bar{y} \in E$. Let $\widetilde{\alpha}$, $\widetilde{\beta} : [0,1] \to E$ be the lifts of the paths, α, $\beta : [0,1] \to B$ with

$$\widetilde{\alpha}(0) = \widetilde{\beta}(0) = e_o,$$

respectively. Choose a path $f : [0,1] \to E$ from $f(0) = e_0$ to $f(1) = \bar{y}$. Let f' be the lift of $p \circ f$ with $f'(0) = \widetilde{\beta}(1)$. Then $\phi([\alpha])(\bar{y}) = f'(1)$. Let $\widetilde{\beta}'$ be the lift of $p \circ \widetilde{\beta} = \beta$ with

$$\widetilde{\beta}'(0) = \widetilde{\alpha}(1).$$

Then $\widetilde{\alpha} \cdot \widetilde{\beta}'$ is the lift of $\alpha\beta : [0,1] \to B$ which starts at e_0. Let f'' be the lift of $p \circ f$ with

$$f''(0) = \widetilde{\alpha} \cdot \widetilde{\beta}'(1) = \widetilde{\beta}'(1).$$

Then, by definition,

$$\phi([\alpha] * [\beta])(\bar{y}) = \phi([\alpha \cdot \beta])(\bar{y}) = f''(1).$$

On the other hand, $\widetilde{\beta} \cdot f''$ is now the lift of $p \circ (\widetilde{\beta} \cdot f')$ with begins at $\widetilde{\alpha}(1)$, so that

$$\phi([\alpha]) \circ \phi([\beta])(\bar{y}) = \phi([\alpha])(f'(1)) = \widetilde{\beta}' \cdot f''(1) = f''(1).$$

Hence,

$$\phi([\alpha] * [\beta]) = \phi([\alpha]) \circ \phi([\beta])$$

and ϕ is indeed a homomorphism.

(d): Let $T \in Cov(E/B)$. Choose any continuous path $\widetilde{\alpha} : [0,1] \to E$ with $\widetilde{\alpha}(0) = e_0$ and $\widetilde{\alpha}(1) = T(e_0)$. Put $\alpha = p \circ \widetilde{\alpha}$. Then

$$\alpha(0) = p \circ \widetilde{\alpha}(0) = p(e_0) = b_0$$

and

$$\alpha(1) = p \circ \widetilde{\alpha}(1) = p(T(e_0)) = p(e_0) = b_0.$$

Therefore, $[\alpha] \in \pi_1(B, b_0)$. In fact, $[\alpha] \in N(H)$. To see why, first recall from Part (a) above that

$$[\alpha]^{-1} p_*(\pi_1(E, e_0))[\alpha] = p_*(\pi_1(E, T(e_0))). \tag{1}$$

On the other hand, since $T : E \to E$ is a homeomorphism, we know that it induces an isomorphism

$$T_* : \pi_1(E, e_0) \to \pi_1(E, T(e_0)).$$

Consequently

$$p_*(\pi_1(E, e_0)) = (p \circ T)_*(\pi_1(E, e_0)) = p_*(T_*(\pi_1(E, e_0))) = p_*(\pi_1(E, T(e_0))). \tag{2}$$

Combining Equations (1) and (2) we get

$$[\alpha]^{-1} p_*(\pi_1(E, e_0))[\alpha] = p_*(\pi_1(E, e_0)),$$

which says that $[\alpha] \in N(H)$.
If now $\bar{y} \in E$ and $f : [0, 1] \to E$ is any path with $f(0) = e_0$ and $f(1) = \bar{y}$, consider $f' = T \circ f$. Since

$$p \circ f' = p \circ (T \circ f) = (p \circ T) \circ f = p \circ f,$$

we see that f' is the lift of $p \circ f$ with $f'(0) = T \circ f(0) = T(e_0) = \widetilde{\alpha}(1)$. Therfore,

$$\phi([\alpha])(\bar{y}) = f'(1) = T(f(1)) = T(\bar{y}).$$

Hence, $T = \phi([\alpha])$, proving that ϕ is onto.

(e): Finally, $\phi([\alpha]) = id_E$ if and only if $f(1) = f'(1)$, which by unique path lifting can only occur when $f = f'$, that is, when $\widetilde{\alpha}(0) = \widetilde{\alpha}(1)$. This is the case precisely when α lifts to a loop at e_0, i.e., when $[\alpha] \in p_*(\pi_1(E, e_0)) = H$. So, the kernel of ϕ equals H.

(iii) Consider the quotient map $q : \mathbb{S}^n \to \mathbb{R}\mathbf{P}^n$. We show that q is a covering projection. Let U_1 be an open subset of $\mathbb{S}^n$ not containing a pair of anti-podal points and

$$U_2 = \{-x/x \in U_1\}.$$

Then, $q(U_1) = q(U_2)$. Denoting these images by U, we see that $q^{-1}(U) = U_1 \cup U_2$ which is an open set in $\mathbb{S}^n$ and so U is open in $\mathbb{R}\mathbf{P}^n$. Second, q maps each of U_1 and U_2 bijectively onto U. To see that q maps each of U_1 and U_2 homeomorphically onto U, we merely have to show that q is an open mapping. So let V_1 be an open subset of U_1 and $V_2 = \{-x | x \in V_1\}$. Then

$$q^{-1}(q(V_1)) = V_1 \cup V_2$$

is open in $\mathbb{S}^n$ so that $q(V_1)$ is an open subset of $\mathbb{R}\mathbf{P}^n$. Thus we have shown that q restricted to each U_j is an open mapping and that suffices for a proof.

We now show that

$$Cov(\mathbb{S}^n/\mathbb{R}\mathbf{P}^n) \cong \mathbb{Z}_2.$$

This follows from part (b) and using the facts that $\pi_1(\mathbb{S}^n) = 1$ and $\pi_1(\mathbb{R}\mathbf{P}^n) = \mathbb{Z}_2$.

(4) (i) Let X be a topolological space, $G \leq Homeo(X)$ a group of homeomorphisms on X. G is said to act properly discontinuous (p.d.) on X if for every $x \in X$, there is a neighborhood, U_x of x, such that $gU_x \cap U_x = \emptyset$ for every $g \in G$ with $g \neq Id_G$.

(ii) We first show that π is an open map. Let $U \subset X$ be open; we need to show $\pi(U)$ is open in X/G. Since X/G has the quotient topology, $\pi(U)$ is open in X/G if and only if $\pi^{-1}(\pi(U))$ is open in X. Following the definition of π, we have

$$\pi^{-1}(\pi(U)) = \bigcup_{g\in G} gU$$

Since each $g \in G$ is a homeomorphism of X, gU is open for every g, so $\pi^{-1}(\pi(U))$ is open in X. We now prove the $\Leftarrow$ direction. The assumption is that G acts p.d. on X. We need to show that π is a covering map. Toward this end, let $[x] \in X/G$ be fixed arbitrarily. Define U to be a neighborhood of x, for some choice of $x \in [x]$, witnessing the p.d.-ness of G's action. The following properties of π are easily checked:

- $\pi(U)$ is a neighborhood of $[x]$.
- $\{gU\}_{g\in G}$ is a disjoint, open decomposition of $\pi^{-1}(\pi(U))$.
- $\pi_{|gU}$ is a continuous, open, surjective map.

It therefore remains to verify that $\pi_{|gU}$ is one-to-one. Toward this end, let x_1, $x_2 \in U$ such that $\pi(gx_1) = \pi(gx_2)$. Then we have $[x_1] = [gx_1] = [gx_2] = [x_2]$ in X/G. So let $g \in G$ such that $x_1 = gx_2$. Since we have $x_1 \in U$ and $gx_2 \in gU$, this implies $U \cap gU$ is nonempty, and thus $g = Id_G$, and therefore that $gx_1 = gx_2$. This concludes the proof of the $\Leftarrow$ direction. To prove the $\Rightarrow$ direction, we are assuming that π is a covering map, and must show that the action of G on X is p.d. Let $x \in X$ be arbitrary, and let V be a neighborhood of $\pi(x)$ which is evenly covered by π. Let $\sqcup_\alpha U_\alpha$ witness this even covering. Let α_0 be such that $x \in U_\alpha$. It is easily derived from the definition of π that, for each α, there is $g \in G$ such that $gU_{\alpha_0} = U_\alpha$. Thus it suffices to show that if $gU_{\alpha_0} \cap U_{\alpha_0}$ is nonempty, then $g = Id_G$. If $y \in gU_{\alpha_0} \cap U_{\alpha_0}$, then we have $\pi(y) = [y] = [g^{-1}y] = \pi(g^{-1}y)$, where y, $g^{-1}y$ are in U_{α_0}. So since $\pi_{|U_{\alpha_0}}$ is a homeomorphism and in particular injective, we have $y = g^{-1}y$. In other words, if $gU_{\alpha_0} \cap U_{\alpha_0}$ is nonempty, then g has a fixed point. But since $\pi = \pi \circ g$ for every $g \in G$, we have that $g \in Cov(X/(X/G))$, so in particular g has no fixed points unless $g = Id_G$. This concludes the proof of the $\Rightarrow$ direction of the if-and-only-if part of the theorem.

We now prove the final two statements of the theorem, starting with the implication that, if π is a covering map, then $G = Cov(X/(X/G))$. We have already shown (in the preceding paragraph) that $G \leq Cov(X/(X/G))$, so it suffices to show that $Cov(X/(X/G)) \leq G$. Toward this end, let $h \in Cov(X/(X/G))$, and let $x \in X$ be any point. Since we have $[h(x)] = \pi(h(x)) = \pi(x) = [x]$, it follows that there is $g \in G$ with $gx = h(x)$. But $g \in Cov(X/(X/G))$, so $g^{-1}h$ fixes x, and thus $g^{-1}h = Id$, so $h = g$, and in particular $h \in G$. Thus $Cov(X/(X/G)) = G$ when π is a covering map.

To prove that π is a regular covering, we know that it suffices to show that G acts transitively on the fibers of π. So let x, $y \in \pi^{-1}([x])$. Then $[x] = [y]$, so $x = gy$ for some $g \in G$, proving transitivity.